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THE INVESTIGATION OF MULTIPLICATION OF SUSPENSION BRIDGE EQUATION USING LINKING THEORY

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Abstract. It is well known that a suspension bridge may display certain oscillations under external aerodynamic forces. Under the action of a strong wind, in particular, a narrow and very flexible suspension bridge can undergo dangerous oscillations. So it would be very contributive to determine under what conditions a similar situation cannot occur, and find out safe parameters of the bridge construction. In this paper, we investigate relations between the multiplicity of solutions and nonlinear terms in this suspension bridge equation using critical point theorem and linking theorem.

1. Introduction

One of the most problematic and not fully explained areas of mathematical modelling involves nonlinear dynamical systems, especially systems with the so called jumping nonlinearity. It can be seen that its presence brings unexpected difficulties into the whole problem and very often it is a cause of multiplicity of solutions. A suspension bridge is an example of such a dynamical system.

The collapse of the Tacoma Narrows suspension bridge caused by a wind blowing at a speed of 42 miles per hour in the State of Washington on November 7, 1940, is one of the most striking examples [2]. So it would be very contributive to determine under what conditions a similar situation cannot occur, and find out safe parameters of the bridge construction.

The nonlinear aspect is caused by the presence of supporting cable stays, which restrain the movement of the center span of the bridge in a

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downward direction, but have no influence on its behavior in the opposite direction. The model describing oscillations in suspension bridge was suggested by McKenna and Walter [7]. The model is described by the nonlinear partial differential equation

$$\begin{aligned} & u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + \kappa u^+ = W(x) + \varepsilon f^*(x,t), \\ & u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \\ & u(x,t+2\pi) = u(x,t), \qquad -\infty < t < \infty, \qquad x \in (0,\pi), \end{aligned} \tag{1}$$

where $\alpha^2 = \frac{EI}{m}(\frac{\pi}{})^4 \neq 0$ and $\beta = \frac{b}{m} > 0$. Under an additional assumption $\alpha = 1$, McKenna and Walter consider the case when the damping term is equal to zero. Then the equation (1) is transformed to the following form :

$$\begin{aligned} & u_{tt} + u_{xxxx} + bu^+ = f(x,t,u), \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ & u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ & \text{u is } \pi\text{-periodic in t and even in x and t,} \end{aligned} \tag{2}$$

where $u^+ = \max\{u, 0\}$.

In this thesis, we investigate the existence of solutions of beam equation for research of the system with wave equation and beam equation. In [1], Nam and Choi consider the asymmetric beam equation where the nonlinear term is a functions with different powers. In that paper, f is defined by

$$f(x,t,s) = \begin{cases} |s|^{p-2}s, & s \geq 0, \\ |s|^{q-2}s, & s < 0, \end{cases}$$

where $p, q > 2$ and $p \neq q$. In this paper, we consider the nonlinear term defined by

$$f(x,t,s) = \begin{cases} |s|^{p-2}s, & s \geq 0, \\ s, & s < 0, \end{cases} \tag{3}$$

where $p > 2$.

In this paper, we use a variational approach and look for critical points of a suitable functional I on a Hilbert space H. Since the functional is strongly indefinite, it is convenient to use the notion of linking. In Section 2, we find a suitable functional I on a Hilbert space H and prove the suitable version of the Palais-Smale condition for the topological method. In Section 3, we study the geometry of the sub-levels of I and find two

linking type inequalities, relative to two different decompositions of the space H.

2. The Palais-Smale condition

To begin with, we consider the associated eigenvalue problem

$$\begin{aligned} &u_{tt} + u_{xxxx} = \lambda u, \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ &u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ &u(x,t) = u(-x,t) = u(x,-t) = u(x,t+\pi). \end{aligned} \tag{4}$$

A simple computation shows that equation (4) has infinitely many eigenvalues λ_{mn} and the corresponding eigenfunctions ϕ_{mn} given by

$$\begin{aligned} &\lambda_{mn} = (2n+1)^4 - 4m^2, \\ &\phi_{mn}(x,t) = \cos 2mt \cos(2n+1)x, \qquad (m,n = 0,1,2,\cdots). \end{aligned}$$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H the Hilbert space defined by

$$H = \{u \in L^2(Q) | u \text{ is even in } x \text{ and } t\}.$$

Then the set $\{\phi_{mn} | m,n = 0,1,2,\cdots\}$ is an orthogonal base of H and H consists of the functions

$$u(x,t) = \sum_{m,n=0}^{\infty} a_{mn}\phi_{mn}(x,t)$$

with the norm given by

$$\|u\|^2 = \int_\Omega u^2(x,t)dxdt.$$

We denote by $(\Lambda_i^-)_{i\geq 1}$ the sequence of the negative eigenvalues of equation (4), by $(\Lambda_i^+)_{i\geq 1}$ the sequence of the positive ones, so that

$$\cdots < \Lambda_1^- = -3 < \Lambda_1^+ = 1 < \Lambda_2^+ = 17 < \cdots .$$

We consider an orthonormal system of eigenfunctions $\{e_i^-, e_i^+, i \geq 1\}$ associated with the eigenvalues $\{\Lambda_i^-, \Lambda_i^+, i \geq 1\}$. We set

$$H^+ = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \geq 0\},$$

$$H^- = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \leq 0\}.$$

We define the linear projections $P^- : H \to H^-, P^+ : H \to H^+$.

We also introduce two linear operators $R : H \to H^+, S : H \to H^-$ by

$$S(u) = \sum_{i=1}^{\infty} \frac{a_i^- e_i^-}{\sqrt{-\Lambda_i^-}}, \qquad R(u) = \sum_{i=1}^{\infty} \frac{a_i^+ e_i^+}{\sqrt{\Lambda_i^+}}$$

if

$$u = \sum_{i=1}^{\infty} a_i^- e_i^- + \sum_{i=1}^{\infty} a_i^+ e_i^+.$$

It is clear that S and R are compact and self adjoint on H.

DEFINITION 2.1. Let $I_b : H \to R$ be defined by

$$I_b(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|[Au]^+\|^2 - \int_\Omega F(Au)dxdt,$$

where $A = R + S$ and $F(s) = \int_0^s f(x,t,\tau)d\tau$.

It is straightforward that

$$\nabla I_b(u) = P^+u - P^-u + bA(Au)^+ - Af(x,t,Au).$$

Following the idea of Hofer (see [5]) one can show that

PROPOSITION 2.2. *$I_b \in C^{1,1}(H,R)$. Moreover $\nabla I_b(u) = 0$ if and only if $w = (R+S)(u)$ is a weak solution of (2), that is,*

$$\int_\Omega (w(v_{tt} + v_{xxxx}) + b[w]^+v)dxdt = \int_\Omega f(x,t,w)vdxdt \text{ for all smooth } v \in H.$$

The following theorem is the uniqueness result for problem (2).

PROPOSITION 2.3. *$b < -\Lambda_1^+$ and*

$$f(x,t,s) = \begin{cases} |s|^{p-2}s, & s \geq 0 \\ 0, & s \leq 0, \end{cases}$$

then problem (2) has only trivial solution.

Proof. Let $Lu = u_{tt} + u_{xxxx}$ and we rewrite (2) as

$$\begin{aligned} Lu - \Lambda_1^+ u &= f(x,t,u) - \Lambda_1^+ u - bu^+ \\ &= (u^+)^{p-1} - \Lambda_1^+ u - bu^+ \\ &= (u^+)^{p-1} - (\Lambda_1^+ + b)u^+ + \Lambda_1^+ u^-. \end{aligned}$$

Multiplying across by e_1^+ and integrating over Ω,

$$\begin{aligned} 0 &= < [L - \Lambda_1^+]u, e_1^+ > \\ &= \int_\Omega [(u^+)^{p-1} - (\Lambda_1^+ + b)u^+ + \Lambda_1^+ u^-]e_1^+ dx \geq 0, \end{aligned}$$

since the condition $b < -\Lambda_1^+$ imply that $-(\Lambda_1^+ + b)u^+ \geq 0$, $(u^+)^{p-1} \geq 0$, and $\Lambda_1^+ u^- \geq 0$ for all real valued function u. and $e_1^+(x) > 0$ for all $(x,t) \in \Omega$. Therefore the only possibility to hold (2) is that $u \equiv 0$. □

REMARK 2.4. $b < -\Lambda_1^+$ and f is defined by equation (3), then problem (2) has no positive solutions.

In this section, we suppose $b > -\Lambda_1^-$. Under this assumption, we have a concern with multiplicity of solutions of equation (2). Here we suppose that f is defined by equation (3).

In the following, we consider the following sequence of subspaces of $L^2(\Omega)$:

$$H_n = (\oplus_{i=1}^n H_{\Lambda_i^-}) \oplus (\oplus_{i=1}^n H_{\Lambda_i^+})$$

where H_Λ is the eigenspace associated to Λ.

LEMMA 2.5. *The functional I_b satisfies $(P.S.)_\gamma^*$ condition, with respect to (H_n), for all γ.*

Proof. Let (k_n) be any sequence in N with $k_n \to \infty$. And let (u_n) be any sequence in H such that $u_n \in H_n$ for all n, $I_b(u_n) \to \gamma$ and $\nabla(I_b) \mid_{H_{k_n}} (u_n) \to 0$.

First, we prove that (u_n) is bounded. By contradiction let $t_n = \|u_n\| \to \infty$ and set $\hat{u_n} = u_n/t_n$. Up to a subsequence $\hat{u_n} \rightharpoonup \hat{u}$ in H for some $\hat{u}$ in H. Moreover

$$\begin{aligned} 0 \;\leftarrow\; & < \nabla(I_b) \mid_{H_{k_n}} (u_n), \hat{u_n} > -\frac{2}{t_n} I_b(u_n) \\ = \;& \frac{2}{t_n}\int_\Omega F(Au_n)dxdt - \frac{1}{t_n}\int_\Omega f(x,t,Au_n)Au_n dxdt \\ = \;& \int_\Omega -\frac{p-2}{p}(t_n)^{p-1}[(A\hat{u_n})^+]^p + 2(t_n)[(A\hat{u_n})^-]^2 dx. \end{aligned}$$

Since $t_n \to \infty$, $(A\hat{u_n})^+ \to 0$ and $(A\hat{u_n})^- \to 0$. This implies $A\hat{u} = 0$ and $\hat{u} = 0$, a contradiction.

So (u_n) is bounded and we can suppose $u_n \rightharpoonup u$ for some $u \in H$. We know that

$$\nabla(I_b) \mid_{H_{k_n}} (u_n) = P^+u_n - P^-u_n + bA(Au_n)^+ - Af(x,t,Au_n).$$

Since A is the compact operator, $P^+u_n - P^-u_n$ converges strongly, hence $u_n \to u$ strongly and $\nabla I_b(u) = 0$. □

3. An Application of Linking Theory

Fixed Λ_i^- and $\Lambda_i^- < -b < \Lambda_{i-1}^-$. We prove the Theorem via a linking argument.

3.1. The First Splitting of Space H. First of all, we introduce a suitable splitting of the space H. Let

$$Z_1 = \oplus_{j=i+1}^{\infty} H_{\Lambda_j^-}, Z_2 = H_{\Lambda_i^-}, Z_3 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-} \oplus H^+$$

LEMMA 3.1. *There exists R such that* $\sup_{v \in Z_1 \oplus Z_2, \|v\|=R} I_b(v) \leq 0$.

Proof. Suppose that $v \in Z_1 \oplus Z_2$. Then we express that $v = \sum_{j=i}^{\infty} a_j^- e_j^-$. Hence $Sv = \sum_{j=i}^{\infty} \frac{a_j^-}{\sqrt{-\Lambda_j^-}} e_j^-$ and $S[Sv] = \sum_{j=i}^{\infty} \frac{a_j^-}{-\Lambda_j^-} e_j^-$. And then

$$I_b(v) = -\frac{1}{2}\|v\|^2 + \frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dxdt.$$

Since $\Lambda_i^- < -b < \Lambda_{i-1}^-$,

$$\begin{aligned} \frac{b}{2}\|[Sv]^+\|^2 \quad &\leq \frac{b}{2}\|Sv\|^2 = \frac{b}{2} < Sv, Sv >= \frac{b}{2} < S[Sv], v > \\ &= \frac{b}{2}\sum_{j=i}^{\infty} \frac{a_j^{-2}}{-\Lambda_j^-} \leq \frac{b}{2}\sum_{j=i}^{\infty} \frac{a_j^{-2}}{-\Lambda_i^-} \leq \frac{b}{2}\sum_{j=i}^{\infty} \frac{a_j^{-2}}{b} = \frac{1}{2}\|v\|^2 \end{aligned}$$

We know that

$$\int_\Omega F(Sv)dxdt = \int_\Omega \left(\frac{1}{p}([Sv]^+)^p + \frac{1}{2}([Sv]^-)^2\right) dxdt.$$

Since $-\frac{1}{2}\|v\|^2 + \frac{b}{2}\|[Sv]^+\|^2 \leq 0$ and $-\int_\Omega F(Sv)dx \leq 0$, there exists R such that $I_b(v) \leq 0$ for all $\|v\| = R$. □

LEMMA 3.2. *There exists $R_1 > R$ such that* $\sup_{v \in Z_1, \|v\| \leq R_1} I_b(v) \leq 0$.

Proof. Repeating the same arguments used in Lemma3.1, we get the conclusion. □

LEMMA 3.3. *There exists ρ such that* $\inf_{u \in Z_2 \oplus Z_3, \|u\|=\rho} I_b(u) > 0$.

Proof. Let $\sigma \in [0, 1]$. We consider the functional $I_{b,\sigma} : Z_2 \oplus Z_3 \to R$ defined by

$$I_{b,\sigma}(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|[Au]^+\|^2 - \sigma\int_\Omega F(Au)dxdt.$$

We claim that there exists a ball $B_\rho = \{u \in Z_2 \oplus Z_3 | \|u\| < \rho\}$ such that

1. $I_{b,\sigma}$ are continuous with respect to σ,
2. $I_{b,\sigma}$ satisfies $(P.S)$ condition,
3. 0 is a minimum for $I_{b,0}$ in B_ρ,
4. 0 is the unique critical point of $I_{b,\sigma}$ in B_ρ.

Then by a continuation argument of Li-Szulkin's (see[6]), it can be shown that 0 is a local minimum for $I_b|_{Z_2 \oplus Z_3} = I_{b,1}$ and Lemma is proved.

The continuity in σ and the fact that 0 is a local minimum for $I_{b,0}$ are straightforward. To prove $(P.S.)$ condition one can argue as in the previous Lemma, when dealing with I_b.

To prove that 0 is isolated we argue by contradiction and suppose that there exists a sequence (σ_n) in $[0,1]$ and sequence (u_n) in $Z_2 \oplus Z_3$ such that $(\sigma_n) \to 0$, $\nabla I_{b,\sigma_n}(u_n) = 0$ for all $n, u_n \neq 0, \text{and} u_n \to 0$. Set $t_n = \|u_n\|$ and $\hat{u_n} = u_n/t_n$ then $t_n \to 0$. Let $\hat{v_n} = P^- \hat{u_n}$ and $\hat{w_n} = P^+ \hat{u_n}$. Since $\hat{v_n}$ varies in a finite dimensional space, we can suppose that $\hat{v_n} \to \hat{v}$ for some $\hat{v}$. We get

$$\frac{1}{t_n}\nabla I_{b,\sigma_n}(u_n) = \hat{w_n} - \hat{v_n} + \frac{b}{t_n}A(Au_n)^+ - \frac{\sigma_n}{t_n}Af(Au_n) = 0. \tag{5}$$

Multiplying by $\hat{w_n}$ yields

$$\|\hat{w_n}\|^2 = \frac{\sigma_n}{t_n}\int_\Omega f(Au_n)A\hat{w_n}dxdt - \frac{b}{t_n}\int_\Omega (Au_n)^+ A\hat{w_n}dxdt.$$

We know that

$$\begin{aligned}\int_\Omega (Au_n)^+ A\hat{w_n}dxdt &= \int_\Omega P^+(Au_n)^+ A\hat{u_n}dxdt \\ &= \int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dxdt.\end{aligned}$$

Since $b > 0$, there exists a sequence (ϵ_n) such that $\epsilon_n \to 0$ and $0 < \epsilon_n < b$ for all n. That is

$$\frac{b}{t_n}\int_\Omega (Au_n)^+ A\hat{w_n}dxdt \geq \frac{\epsilon_n}{t_n}\int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dxdt.$$

Then

$$\begin{aligned}\|\hat{w_n}\|^2 &\leq \frac{\sigma_n}{t_n}\int_\Omega f(Au_n)A\hat{w_n}dxdt - \frac{\epsilon_n}{t_n}\int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dxdt \\ &\leq \sigma_n\int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{w_n}|dxdt + \epsilon_n\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dxdt.\end{aligned}$$

Since A is a compact operator

$$\begin{aligned} |f(Au_n)| &= |\{([t_nA\hat{u_n}]^+)^{p-1} - ([t_nA\hat{u_n}]^-)\}| \\ &\leq {t_n}^{p-1}|[A\hat{u_n}]^+|^{p-1} + t_n|[A\hat{u_n}]^-| \\ &\leq t_n(M_1 + {t_n}^{p-2}M_2) \end{aligned}$$

for some M_1 and M_2. We get that

$$\frac{\sigma_n}{t_n}|f(Au_n)| \leq \frac{\sigma_n}{t_n}t_n(M_1 + {t_n}^{p-2}M_2) \leq \sigma_n(M_1 + {t_n}^{p-2}M_2) \leq o(1),$$

and

$$\sigma_n \int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{w}_n|dxdt \leq \sigma_n(M_1 + {t_n}^{p-2}M_2)\int_\Omega |A\hat{w}_n|dxdt \leq o(1).$$

Hence

$$\|\hat{w_n}\|^2 \leq o(1) + \epsilon_n \int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dxdt. \tag{6}$$

Since $\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dxdt$ is bounded and equation (6) holds $\hat{w_n} \to 0$ and so $(\hat{u_n})$ converges. Since $\frac{\sigma_n}{t_n}|f(Au_n)| \leq o(1)$, we get $\frac{\sigma_n}{t_n}Af(Au_n) \to 0$. From equation (5), $-\hat{v} + bA(A\hat{v})^+ = 0$ and so $\hat{v} = bA(A\hat{v})^+ \geq 0$. Thus $(\hat{v})^- = 0$ and $(\hat{v})^+ = bA(A\hat{v})^+$. Multiplying by $e_j^-(1 \leq j \leq i)$ yields

$$\begin{aligned} a_i^- &=< \hat{v}, e_j^- >= b\int_\Omega (A\hat{v})^+(Ae_j^-)dxdt \\ &= \frac{b}{\sqrt{-\Lambda_j^-}}\int_\Omega (A\hat{v})^+(e_j^-)dxdt = \frac{b}{-\Lambda_j^-}a_i^- \end{aligned}$$

and so $a_j^-(\frac{b}{-\Lambda_j^-} - 1) = 0$. Since $b \neq -\Lambda_j^-$, $a_j^- = 0$ for all $j = 1, 2, \cdots$ and so $(\hat{v_n})$ converges to zero, but this is impossible if $\|\hat{(u_n)}\| = 1$. □

Definition 3.4. Let H be an Hilbert space, $Y \subset H$, $\rho > 0$ and $e \in H \setminus Y$, $e \neq 0$. Set:

$$\begin{aligned} B_\rho(Y) &= \{x \in Y \mid \|x\| \leq \rho\}, \\ S_\rho(Y) &= \{x \in Y \mid \|x\| = \rho\}, \\ \triangle_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| \leq \rho\}, \\ \Sigma_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| = \rho\} \cup \{v \mid v \in Y, \|v\| \leq \rho\}. \end{aligned}$$

THEOREM 3.5. *If $\Lambda_i^- < -b < \Lambda_{i-1}^-$ then problem (2) has at least one nontrivial solution.*

Proof. Let $e \in Z_2$. By Lemma 3.1, Lemma 3.2, and Lemma 3.3, for a suitable large R and a suitable small ρ, we have the linking inequality

$$\sup I_b(\Sigma_R(e, Z_1)) < \inf I_b(S_\rho(Z_2 \oplus Z_3)). \tag{7}$$

Moreover $(P.S.)^*_\gamma$ holds. By standard linking arguments, it follows that there exists a critical point u for I_b with $\alpha \le I_b(u) \le \beta$, where $\alpha = \inf I_b(S_\rho(Z_2 \oplus Z_3))$ and $\beta = \sup I_b(\Delta_R(e, Z_1))$. Since $\alpha > 0$, then $u \neq 0$. □

3.2. The Second Splitting of Space H. We assume in this section that $i \ge 2$ and we set

$$W_1 = \oplus_{j=i}^{\infty} H_{\Lambda_j^-}, W_2 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-}, W_3 = H^+.$$

Notice that $W_1 = Z_1 \oplus Z_2$ and $W_2 \oplus W_3 = Z_3$.

LEMMA 3.6. $\liminf_{\|u\| \to +\infty, u \in W_1 \oplus W_2} I_b(u) \le 0$.

Proof. Let $(u_n)_n$ be a sequence in $W_1 \oplus W_2$ such that $\|u_n\| \to \infty$. We set $t_n = \|u_n\|$ and $\hat{u_n} = u_n/t_n$. Since S is a compact operator,

$$\begin{aligned} \frac{b}{2}\frac{\|[Su_n]^+\|^2}{t_n^2} &- \int_\Omega \frac{F(Su_n)}{t_n^2} dxdt \\ &= \int_\Omega \frac{b}{2}([S\hat{u_n}]^+)^2 - \frac{{t_n}^{p-2}}{p}([S\hat{u_n}]^+)^p - \frac{1}{2}([S\hat{u_n}]^-)^2 dxdt \\ &\to -\infty. \end{aligned}$$

Then

$$\frac{I_b(u_n)}{\|u_n\|^2} = -\frac{1}{2} + \frac{b}{2}\frac{\|[Su_n]^+\|^2}{t_n^2} - \int_\Omega \frac{F(Su_n)}{t_n^2} dxdt \to -\infty.$$

Hence

$$\liminf_{\|u\| \to +\infty, u \in W_1 \oplus W_2} I_b(u) \le 0.$$

□

LEMMA 3.7. *There exists $\hat{\rho}$ such that* $\inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) > 0$.

Proof. Repeating the same arguments used in Lemma 3.2, we get the conclusion. □

THEOREM 3.8. *Let $i \geq 2$. If $\Lambda_i^- < -b < \Lambda_{i-1}^-$ then problem (2) has at least two nontrivial solution.*

Proof. Using the conclusion of Theorem 3.5, we have that there exist a nontrivial critical point u with

$$I_b(u) \leq \sup I_b(\Delta_R(e, Z_1))$$

where R, R_1, e were given in Lemma 3.1, Lemma 3.2 and 3.5. We can choose that $R_1 \geq \hat{R} \geq R$. Take any $\hat{e}$ in W_2, then we have a second linking inequality,

$$\sup I_b(\Sigma_{\hat{R}}(\hat{e}, W_1)) \leq \inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)).$$

Since $(P.S.)^*_\gamma$ holds, there exists a critical point $\hat{u}$ such that

$$\inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) \leq I_b(\hat{u}) \leq \sup I_b(\Delta_{\hat{R}}(\hat{e}, W_1)).$$

Since $\hat{R} \geq R$ and $Z_1 \oplus Z_2 = W_1$,

$$\Delta_R(e, Z_1) \subset B_{\hat{R}}(W_1) \subset \Sigma_{\hat{R}}(\hat{e}, W_1).$$

Then

$$\begin{aligned} I_b(u) &\leq \sup I_b(\Delta_R(e, Z_1)) \\ &\leq \sup I_b(\Sigma_{\hat{R}}(\hat{e}, W_1)) < \inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) \leq I_b(\hat{u}). \end{aligned}$$

Hence $u \neq \hat{u}$. □

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